On the structure of the new electromagnetic conservation laws.

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Abstract. New electromagnetic conservation laws have recently been proposed: in the absence of electromagnetic currents, the trace of the Chevreton superenergy tensor, H_{ab} is divergence-free in four-dimensional (a) Einstein spacetimes for test fields, (b) Einstein-Maxwell spacetimes. Subsequently it has been pointed out, in analogy with flat spaces, that for Einstein spacetimes the trace of the Chevreton superenergy tensor H_{ab} can be rearranged in the form of a generalised wave operator $\Box_L$ acting on the energy momentum tensor T_{ab} of the test fields, i.e., $H_{ab} = \Box_L T_{ab}/2$. In this letter we show, for Einstein-Maxwell spacetimes in the full non-linear theory, that, although, the trace of the Chevreton superenergy tensor H_{ab} can again be rearranged in the form of a generalised wave operator $\Box_G$ acting on the electromagnetic energy momentum tensor, in this case the result is also crucially dependent on Einstein's equations; hence we argue that the divergence-free property of the tensor $H_{ab} = \Box_G T_{ab}/2$ has significant independent content beyond that of the divergence-free property of T_{ab}.

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It is easy to see that the energy-momentum tensor of source-free electromagnetic fields (notation and conventions from [1])

$$T_{ab} = F_{ac}F_b{}^c - g_{ab}F_{cd}F^{cd}/4 \tag{1}$$

is divergence-free as a consequence of the source-free Maxwell equations in *arbitrary n-dimensional spaces.* Bergqvist, Eriksson and Senovilla (BES) [2] have recently found that the trace of the Chevreton tensor [3],[4],

$$H_{ab} \equiv \nabla^e F_{ac}\nabla_e F_b{}^c - g_{ab}\nabla^e F_{cd}\nabla_e F^{cd}/4 \tag{2}$$

is divergence-free in *four* dimensions for:
(a) *Einstein spaces*, with the source-free electromagnetic field considered as a test field so that the geometry of the space is unaffected;
(b) *source-free Einstein-Maxwell spaces* (including a possible cosmological constant).
Subsequently, Deser [5] has rederived (the Ricci-flat space version of) part (a) of the BES result by showing, in *four-dimensional Ricci-flat spaces* that H_{ab} can be rearranged into the form

$$2H_{ab} = \Box_L T_{ab} \equiv \nabla_i\nabla^i T_{ab} - 2C_{aibj}T^{ij} \ , \tag{3}$$

and then exploiting the commutativity property

$$\big[\Box_L, \nabla^b\big]X_{ab} = 0 \tag{4}$$

which is valid for *arbitrary* symmetric 2-index tensors X_{ab}, where $\Box_L$ is the Lichnerowicz generalised wave operator [6] in Ricci-flat spaces†; from this it follows that $\Box_L T_{ab}$ is divergence-free, $\nabla^b\big(\Box_L T_{ab}\big) = 0$. This shows that part (a) of the BES result can be seen as a consequence of the fact that T_{ab} is divergence-free, combined with the commutativity property (4).

† Lichnerowicz [6] has defined a generalised wave operator acting on an arbitrary tensor in arbitrary spaces. In

An obvious question is whether such a rearrangement as (3) and a commutator property such as (4) also exist in spaces more general than Ricci-flat spaces. In this letter we show that, with an alternative generalised wave operator $\Box_G$, we can obtain an analogous — and we argue deeper — result for *Einstein-Maxwell spaces.*

We consider directly H_{ab} from (2), and rearrange as follows:

$$\begin{aligned}
2H_{ab} =& \Big(\Box\big(F_{ac}F_b{}^c\big) - F_{ac}\Box F_b{}^c - F_b{}^c\Box F_{ac} - \frac{1}{4}g_{ab}\Box\big(F_{cd}F^{cd}\big) + \frac{1}{2}g_{ab}F_{cd}\Box F^{cd}\Big)\\
=& \Box\Big(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\Big)\\
&- F_a{}^c\Big(2C_{ijbc}F^{ij} + 2\frac{n-4}{n-2}F_{i[b}\tilde{R}_{c]}{}^i - \frac{2(n-2)}{n(n-1)}F_{bc}R\Big)\\
&- F_b{}^c\Big(2C_{ijac}F^{ij} + 2\frac{n-4}{n-2}F_{i[a}\tilde{R}_{c]}{}^i - \frac{2(n-2)}{n(n-1)}F_{ac}R\Big)\\
&+ \frac{1}{2}g_{ab}F^{cd}\Big(2C_{icjd}F^{ij} + 2\frac{n-4}{n-2}F_{i[c}\tilde{R}_{d]}{}^i - \frac{2(n-2)}{n(n-1)}F_{cd}R\Big)
\end{aligned} \tag{5}$$

having made use of the result which follows from the the source-free Maxwell's equations, that [7]

$$\Box F_{ab} = 2C_{ijab}F^{ij} + 2\frac{n-4}{n-2}F_{i[a}\tilde{R}_{b]}{}^i - \frac{2(n-2)}{n(n-1)}F_{ab}R \tag{6}$$

where $\tilde{R}_{ab}(\equiv R_{ab} - Rg_{ab}/n)$ is the trace-free part of the Ricci tensor R_{ab}.
When we specialise to *four* dimensions, not only do the terms with the trace-free Ricci tensor disappear identically in (5), but we can exploit the *four-dimensional identity*

$$C_{[ab}{}^{[de}\delta_{c]}^{f]} = 0 \tag{7}$$

to obtain the identity

$$0 = 9F^{ij}F_{kl}C_{[ai}{}^{[bk}\delta_{j]}^{l]} = -2C_{ij}{}^b{}_cF^{ij}F_a{}^c - 2C_{ijac}F^{ij}F^{bc} + \delta_a^bF^{cd}C_{ijcd}F^{ij} + 4C_{ai}{}^{bk}F^{ij}F_{kj} \tag{8}$$

and hence (5) becomes, in *four-dimensional spaces*,

$$2H_{ab} = \Box T_{ab} - 2T_{ij}C_a{}^i{}_b{}^j + \frac{2R}{3}T_{ab}. \tag{9}$$

Note that in calculating (9) we have used the source-free Maxwell equations and the definition of the electromagnetic energy-momentum tensor in (1), but not Einstein's equations.
Although this expression for H_{ab} looks similar to the expression (3) exploited by Deser [5] and written in terms of the Lichnerowicz operator $\Box_L$ for Ricci-flat spaces, this version (9) does not coincide with $\Box_L T_{ab}/2$ for *arbitrary* spaces.

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particular, for *arbitrary* symmetric 2-index tensors in *arbitrary spaces* it has the form $\Delta X_{ab} \equiv -\nabla_i\nabla^i X_{ab} - R_{ai}X^i{}_b - R_{bi}X^i{}_a + 2R_{aibj}X^{ij}$; Lichnerowicz has shown that in spaces satisfying $\nabla_c R_{ab} = 0$ there exists the identity

$$\big[\Delta, \nabla^b\big]X_{ab} = 0$$

where $\Delta Y_a \equiv -\nabla_i\nabla^i Y_a - R_{ai}Y^i$. We shall reverse the sign and define $\Box_L X_{ab} \equiv \nabla_i\nabla^i X_{ab} + R_{ai}X^i{}_b + R_{bi}X^i{}_a - 2R_{aibj}X^{ij}$ which agrees with the version used by Deser [5] in Ricci-flat spaces and written as $L(T_{ab}) \equiv \nabla_i\nabla^i T_{ab} - 2R_{aibj}T^{ij}$. (In [5] and [6] a different sign convention is used for the Riemann tensor than we use in this paper, following [1], so in the counterparts to these expressions in [5], [6], the terms involving Riemann tensors have a sign difference.)

For *Ricci-flat spaces*, the divergence of $\Box_L X_{ab}$ (or, in particular, of $H_{ab} = \Box_L T_{ab}/2$) follows directly from the divergence of arbitrary X_{ab} (or, in particular, of T_{ab}). On the other hand, for the full non-linear case, when we calculate the divergence of H_{ab} in (9), we find that we need to use explicitly the fact that T_{ab} *is a divergence-free tensor which is equivalent to the Einstein tensor via Einstein's equations.* We could show this by a direct calculation of the divergence of H_{ab} from (9), but it will be more instructive to exploit a commutator property, motivated by the special commutator identity in [5] for n-dimensional Ricci-flat spaces. Any symmetric 2-tensor X_{ab} is easily seen to satisfy

$$\Box \nabla^b X_{ab} = \nabla^b \Big(\Box X_{ab} + R_b{}^i X_{ai} - 2R_a{}^i{}_b{}^j X_{ij} \Big) - 2\nabla_{[j} R_{a]i} X^{ij} \ , \tag{10a}$$

or equivalently

$$\Big(\delta_a^i \Box - R_a{}^i \Big) \nabla^b X_{ib} = \nabla^b \Big(\Box X_{ab} - 2R^i{}_{[a} X_{b]i} - 2R_a{}^i{}_b{}^j X_{ij} \Big) + \nabla_a R^{ij} X_{ij} \ . \tag{10b}$$

Note that in arbitrary spaces we cannot deduce the existence of divergence-free tensors involving second derivatives of an arbitrary tensor X_{ab} from either (10a) or (10b); specialising X_{ab} to be divergence-free but otherwise arbitrary does not help. On the other hand, there are cases where the existence of such tensors can be deduced by either restricting the background spaces (e.g. Ricci-flat spaces as in [5]), and/or restricting the arbitrariness of the tensor X_{ab}.
If we now replace the arbitrary tensor X_{ab} with the Einstein tensor $G_{ab} (\equiv R_{ab} - Rg_{ab}/2)$ we obtain, for *all n-dimensional spaces*,

$$\begin{aligned} \Box_G \nabla^b G_{ab} &= \nabla^b \Big(\Box G_{ab} - 2R_{i[a} G^i{}_{b]} - 2R_{aibj} G^{ij} \Big) + \nabla_a R_{ij} G^{ij} \\ &= \nabla^b \Big(\Box G_{ab} - 2R_{aibj} G^{ij} + \frac{1}{2} g_{ab} G_{ij} R^{ij} \Big) \end{aligned} \tag{11}$$

which yields the commutator identity

$$\Big[\nabla^b, \ \Box_G \Big] G_{ab} = 0 \tag{12}$$

where $\Box_G G_{ab} \equiv \Box G_{ab} - 2R_a{}^i{}_b{}^j G_{ij} + g_{ab} G_{ij} R^{ij}/2$ and $\Box_G \big(\nabla^b T_{ab} \big) \equiv \Box \big(\nabla^b T_{ab} \big) - R^{ai} \nabla^b T_{ab}$; as a consequence of the divergence-free property of G_{ab} the symmetric tensor $\Box_G G_{ab}$ is divergence-free.
We note that this tensor $\Box_G G_{ab}$ is a purely geometric construction which is divergence-free in *all n*-dimensional spaces; it has previously been identified in a number of different papers [8].
Using Einsteins's equations ($G_{ab} = -\kappa T_{ab}$) to replace the Einstein tensor with the energy-momentum tensor, and specialising to four dimensions, we obtain the divergence-free tensor

$$\Box_G T_{ab} = \Box T_{ab} - 2C_{aibj} T^{ij} - 2\kappa \Big(T_{ia} T^i{}_b - \frac{1}{4} T^{ij} T_{ij} g_{ab} \Big) + \kappa \frac{2T}{3} \Big(T_{ab} - T g_{ab}/4 \Big) \ . \tag{13}$$

A further specialisation to electromagnetic fields means that the penultimate term in (13) disappears when we use the four-dimensional algebraic Rainich conditions [9], and the final term disappears when we substitute $T = 0$, resulting in the divergence-free tensor,

$$\Box_G T_{ab} = \Box T_{ab} - 2C_{aibj} T^{ij} \tag{14}$$

When we apply Einstein's equations in (9) (substituting R=0), and compare with equation (14), we obtain confirmation of part (b) of the BES result, and in particular that $H_{ab} = \Box_G T_{ab}/2$ is divergence-free in Einstein-Maxwell spaces in four dimensions. (In [2] the result for Einstein-Maxwell spaces was shown to be valid even for the case where Einstein's equations had a non-zero cosmological constant; this can easily be confirmed by comparing the coefficients of the T terms in (9) and (14) respectively.)

The original BES results were obtained in spinors, and the authors pointed out that a tensor derivation was 'far from obvious'. The inbuilt four-dimensional simplicity for spinors has its counterpart in the much more complicated exploitation of four-dimensional identities in tensors [10]. In the above analysis it is clear that

the four-dimensional identity (7) plays a crucial role in the tensor proof of part (b) of the BES result (as it also does in part (a)); in addition the four-dimensional Rainich identity is essential to simplify (5) in the proof of part (b). For *higher dimensions* $n > 4$, it is easy to see that the terms involving the trace-free Ricci tensor $\tilde{R}_{ab}$ do not disappear in (5), and it is straightforward to show that there are no analogous identities which would enable the quadratic terms in F_{ab} on the right hand side of (5) to be replaced with T_{ab}, nor are there any appropriate higher dimensional Rainich identities. Hence the divergence-free property of H_{ab} is restricted to *four* dimensions in Einstein-Maxwell spaces, as is the identity (9).

These calculations demonstrate formally, as in flat and Ricci-flat spaces, that the divergence-free nature of the energy-momentum tensor T_{ab} leads to the divergence-free property of an expression which can be considered as a generalised wave operator on T_{ab}. In the flat space case Deser [5] argued that this expression supplied no independent content, and the same criticism might be made of the Ricci-flat space case — in both of these cases the divergence of *any* tensor X_{ab} leads *directly* to the divergence of a generalised wave operator on X_{ab}. However, in Einstein-Maxwell spaces there is a significant new input required to ensure the divergence of $H_{ab} = \Box_G T_{ab}$ — Einstein's equations are crucial. A confirmation of the fact that the divergence-free property of H_{ab} brings something new is given by Senovilla's demonstration that non-trivial divergence-free currents can be constructed from H_{ab} in situations where the divergence-free currents constructed from T_{ab} are trivial [11].
Although the general symmetric divergence-free tensor $\Box_G G_{ab}$ constructed from the divergence-free Einstein tensor G_{ab} has been quoted in a number of places in the literature [8] (usually as an aside in the investigations of 4-index tensors of the Bel-Robinson type [12]), there seems to have been no significant investigation of the energy-momentum counterpart $\Box_G T_{ab}$ obtained via Einstein's equations. The significance and usefulness of this tensor and the corresponding divergence-free currents need to be explored.

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